

Free streaming in mixed dark matter

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Free streaming in a *mixture* of collisionless non-relativistic dark matter (DM) particles is studied by solving the linearized Vlasov equation implementing methods from the theory of multicomponent plasmas. The mixture includes Fermionic, condensed and non-condensed Bosonic particles decoupling in equilibrium while relativistic, heavy thermal relics that decoupled when non-relativistic (WIMPs), and sterile neutrinos that decouple *out of equilibrium* when they are relativistic. The different components interact via the self-consistent gravitational potential that they source. The free-streaming length λ_{fs} is obtained from the marginal zero of the gravitational polarization function, which separates short wavelength Landau-damped from long wavelength Jeans-unstable *collective* modes. At redshift z we find $\frac{1}{\lambda_{fs}^2(z)} = \frac{1}{(1+z)} \left[\frac{0.071}{\text{kpc}} \right]^2 \sum_a \nu_a g_{d,a}^{\frac{2}{3}} (m_a/\text{keV})^2 I_a$, where $0 \leq \nu_a \leq 1$ are the *fractions* of the respective DM components of mass m_a that decouple when the effective number of ultrarelativistic degrees of freedom is $g_{d,a}$, and I_a are dimensionless ratios of integrals of the distribution functions which only depend on the microphysics at decoupling and are obtained explicitly in all the cases considered. If sterile neutrinos produced either resonantly or non-resonantly that decouple near the QCD scale are the *only* DM component, we find $\lambda_{fs}(0) \simeq 7 \text{ kpc (keV/m)}$ (non-resonant), $\lambda_{fs}(0) \simeq 1.73 \text{ kpc (keV/m)}$ (resonant). If WIMPs with $m_{wimp} \gtrsim 100 \text{ GeV}$ decoupling at $T_d \gtrsim 10 \text{ MeV}$ are present in the mixture with $\nu_{wimp} \gg 10^{-12}$, $\lambda_{fs}(0) \lesssim 6.5 \times 10^{-3} \text{ pc}$ is *dominated* by CDM. If a Bose Einstein condensate is a DM component its free streaming length is consistent with CDM because of the infrared enhancement of the distribution function.

I. INTRODUCTION

Candidate dark matter (DM) particles are broadly characterized as cold, hot or warm depending on their velocity dispersions. The *concordance* Λ CDM standard cosmological model emerging from CMB, large scale structure observations and simulations favors the hypothesis that DM is composed of primordial particles which are cold and collisionless[1]. Although this model is very successful in describing the large scale distribution of galaxies, recent observations hint at possible discrepancies summarized as the “satellite” and “cuspy halo” problems. In the Λ CDM model the CDM power spectrum favors small scales, which become non-linear first and collapse in a hierarchical “bottom-up” manner and dense clumps survive the mergers in the form of “satellite” galaxies. Large-scale simulations within the Λ CDM paradigm lead to an overprediction of “satellite” galaxies [2], which is almost an order of magnitude larger than the number of satellites that have been observed in Milky-Way sized galaxies[2, 3, 4, 5, 6].

Furthermore, large-scale N-body simulations of CDM clustering predict a density profile monotonically increasing towards the center of the halos[2, 7, 8, 9, 10], with asymptotic behavior $\rho(r) \sim r^{-\gamma}$ where $1 \leq \gamma \lesssim 1.5$ [4, 8, 10] which describes accurately clusters of galaxies, but indicates a divergent cusp at the center of the halo. In contrast with these results, recent observations seem to indicate central cores in dwarf galaxies[11, 12, 13, 14, 15], sparking the “cusps vs cores” controversy.

Warm dark matter candidates (WDM) have been invoked as possible solutions to these potential discrepancies[16, 17], these particles feature velocity dispersions intermediate between CDM and HDM and can relieve the satellite and cuspy halo problems. The clustering properties of collisionless DM in the linear regime depend on a fundamental length scale: the free-streaming length, that determines a cutoff in the power spectrum. Length scales larger than the free-streaming scale undergo gravitational instability, and shorter scales are damped. A *simple estimate* of the free-streaming length λ_{fs} is obtained from the familiar Jeans’ length by replacing the speed of sound by the velocity dispersion of the particle. An equivalent estimate is obtained by computing the distance that the particle travels within a dynamical (Hubble) time[18] $\lambda_{fs} \sim \langle \vec{V}^2 \rangle^{\frac{1}{2}} / H_0$.

However, a thorough assessment of DM particles and structure formation requires a more detailed and reliable determination of the free-streaming length. The necessity for this has been recently highlighted by the recent results of ref.[19] that suggest that the first stars form in filaments of the order of the free streaming scale.

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Perturbations in a collisionless system of particles with gravitational interactions is fundamentally different from fluid perturbations in the presence of gravity. The (perfect) fluid equations correspond to the limit of vanishing mean free path. In a gravitating fluid pressure gradients tend to restore hydrostatic equilibrium with the speed of sound in the medium and short wavelength fluctuations are simple acoustic waves. For large wavelengths the propagation of pressure waves cannot halt gravitational collapse on a dynamical time scale. The dividing line is the Jeans length: perturbations with wavelengths shorter than this oscillate as sound waves, while perturbations with longer wavelength undergo gravitational collapse.

In a gas of collisionless particles with gravitational interaction the situation is different since the mean free path is much larger than the size of the system (Hubble radius) and the fluid description is not valid. Instead the Boltzmann-Vlasov equation for the distribution function must be solved to extract the dynamics of perturbations[20, 21]. Just as in the case of plasma physics, the linearized Boltzmann-Vlasov equation describes *collective* excitations[22]. In the case of a collisionless gas with gravitational interactions these collective excitations describe particles free-streaming in and out of the gravitational potential wells of which they are the source. The damping of short wavelength *collective* excitations is akin to *Landau damping in plasmas*[22] and is a result of the phenomenon of *dephasing* via phase mixing in which the particles are out of phase with the potential wells that they produce[23]. This situation is similar to Landau damping in plasmas where dephasing between the charged particles and the self-consistent electric field that they produce lead to the collisionless damping of the collective modes[22]. For a thorough review of collective excitations and their Landau damping in gravitational systems see ref.[24].

Gilbert[25] studied the linearized Boltzmann-Vlasov equation in a matter dominated cosmology for non-relativistic particles described by an (unperturbed) Maxwell-Boltzmann distribution function. In this reference the linearized this equation was cast as a Volterra integral equation of the second kind which was solved numerically. The result of the integration reveals a limiting value of the wavevector below which perturbations are Landau damped and above which perturbations grow via gravitational instability. Eventually the redshift in the expanding cosmology makes wavevectors that are initially damped to enter the band of unstable modes and grow[25]. However, the dividing wavevector between damped and growing modes emerges at very early times during which the expansion can be neglected[25]. The results of the numerical study were consistent with replacing the speed of sound by the Maxwellian velocity dispersion in the Jeans length (up to a normalization factor of $\mathcal{O}(1)$). Gilbert's equations were used also by Bond and Szalay[26] in their pioneering study of collisionless damping of density fluctuations in an expanding cosmology. These authors focused primarily on massive neutrinos and solved numerically Gilbert's integral equation but approximated the Fermi-Dirac distribution function and provided a fitting function for the numerical results. Gilbert's equations were also solved numerically to study dissipationless clustering of neutrinos in ref.[27] but with a truncation of their Fermi-Dirac distribution function and an analytic fit to the numerical solution of the integral equation was provided. Bertschinger and Watts[21, 28] also studied numerically Gilbert's equation within the context of cosmological perturbations from cosmic strings and massive light neutrinos, and more recently similar integral equations were solved approximately for thermal neutrino relics in ref.[29].

Most of the studies of the solutions of the linearized Boltzmann-Vlasov equation for collisionless particles addressed one single species of DM candidates¹ and generally in terms equilibrium distribution functions or approximations thereof, for example Maxwell-Boltzmann for relics that decouple non-relativistically or Fermi-Dirac (without chemical potential) (or truncations of this distribution) for neutrinos.

However, it is likely that DM may be composed of *several* species, this possibility is suggested in ref.[30] and most extensions of the standard model generally allow several possible candidates, from massive weakly interacting particles (WIMPs) to “sterile” neutrinos ($SU(2)$ singlets)[31].

Furthermore, several possible WDM candidates may decouple *out of local thermodynamic equilibrium* (LTE) with distribution functions which may be very different from the usual ones in LTE. This is the case for sterile neutrinos produced non-resonantly via the Dodelson-Widrow (DW) mechanism[32] or through a lepton-driven MSW (Mikheyev-Smirnov-Wolfenstein) resonance[33].

A. Motivation and goals: In this article we study free streaming of decoupled collisionless non-relativistic (DM) candidates focusing on *two* aspects:

- A *mixture* of (DM) particles including CDM and WDM candidates: typical studies of structure formation invoke either CDM *or* WDM, but it is likely that *both* candidates are present with different fractions ν of the total DM component. In fact most particle physics extensions beyond the standard model have plenty of room for a variety of CDM, WDM or HDM candidates. Thus we allow all of these components, each one contributing an

¹ The exception being ref.[28] wherein massive neutrinos decoupled with a Fermi-Dirac distribution and a cosmic string source $\propto \delta^3(\vec{r})$ were studied, neglecting any radiation component.

arbitrary fraction ν to the total (DM) content of the Universe. Although the DM candidates are collisionless and do not interact directly with each other, they interact *indirectly* via the gravitational potential that they source. As a result the free-streaming length of the mixture is a non-trivial function of the individual free-streaming lengths.

- Free streaming has mostly been studied in the above references with particles that decoupled either when ultrarelativistic (as is the case for neutrinos) or non-relativistic as in the case of weakly interacting massive particles (WIMPs) but generally in local thermodynamic equilibrium (LTE), namely with Fermi-Dirac, Bose-Einstein or Maxwell-Boltzmann distributions respectively. We seek to obtain the corresponding free streaming lengths for particles that decoupled *in or out* of LTE with *arbitrary* isotropic distribution functions, without any truncation. This aspect is important for sterile neutrinos either produced non-resonantly[32] or resonantly[31, 33] because these particles decoupled while relativistic but *out* of LTE. Therefore, we consider the most general *mixture* of Fermionic and Bosonic thermal relics that decouple when relativistic, including the possibility of a Bose-Einstein condensate (BEC)[34], heavy non-relativistic thermal relics, as the case of WIMPs, and sterile neutrinos that decoupled *out of LTE* when ultrarelativistic.

In order to carry out this program analytically we first neglect the cosmological expansion and solve the linearized Boltzmann-Vlasov equation in the non-expanding case *exactly* by implementing methods from the theory of multi-component plasmas[22]. The neglect of the cosmological expansion is warranted by the detailed numerical study in refs.[25, 26] wherein it was found that the dividing wavevector between the Landau damped modes and the modes that grow under gravitational instability is insensitive to the expansion, just as in the case of the Jeans instability where the Jeans wavevector can be extracted in the non-expanding case include the redshift dependence of the density, speed of sound and scale factors *a posteriori*[18]. The analytic *exact* solution for the free-streaming wave-vector *today* $k_{fs}(0) = 2\pi/\lambda_{fs}(0)$ in terms of the full distribution functions without truncation, *in or out of LTE* is a main result of this program, one which yields a reliable determination of free-streaming lengths for mixtures of (DM) components that decoupled with arbitrary distribution functions.

This program is carried out by implementing methods from the theory of multicomponent plasmas[22], in particular we obtain the “gravitational polarization” function[23, 24] for a *mixture* of (DM) components akin to the dielectric response function of multicomponent plasmas[22]. The collective excitations are described by the zeroes of this function in the complex frequency plane and the free-streaming wave-vector k_{fs} is identified as that wavevector that separates between the Landau-damped short wavelength modes and the gravitationally Jeans- unstable long-wavelength modes.

Based on the Liouville evolution of the decoupled distribution functions and assuming that the expansion of the universe is slow enough so that it can be treated adiabatically, we provide a scaling argument that determines the following dependence of the free streaming length on the redshift,

$$\lambda_{fs}(z) = \lambda_{fs}(0)\sqrt{1+z}. \quad (1.1)$$

B. Summary of Results: Our main result for the comoving free streaming length $\lambda_{fs}(z)$ at redshift z of mixed (DM) is

$$\frac{1}{\lambda_{fs}^2(z)} = \frac{1}{(1+z)} \left[\frac{0.071}{\text{kpc}} \right]^2 \sum_{\text{species}} \left\{ \nu_F g_{d,F}^{\frac{2}{3}} \left(\frac{m_F}{\text{keV}} \right)^2 I_F[u] + \nu_s g_{d,s}^{\frac{2}{3}} \left(\frac{m_s}{\text{keV}} \right)^2 6.814 + \nu_B g_{d,B}^{\frac{2}{3}} \left(\frac{m_B}{\text{keV}} \right)^2 I_B[x_d, u_d] + 10^{12} \nu_{wimp} g_{d,wimp}^{\frac{2}{3}} \left(\frac{m_{wimp}}{100 \text{ GeV}} \right) \left(\frac{T_d}{10 \text{ MeV}} \right) \right\}, \quad (1.2)$$

where ν_a is the *fraction* of (DM) of each component with $\sum_a \nu_a = 1$, $g_{d,a}$ is the effective number of ultrarelativistic degrees of freedom at decoupling for each species (a) of mass m_a , and the functions I_F, I_B are dimensionless ratios of integrals of the distribution functions of the decoupled particles which are determined by the microphysics at decoupling. Their explicit expressions in the cases considered are given in section (III). The label F refer to *all* possible Fermions with chemical potential μ decoupled in LTE at a temperature T_d while ultrarelativistic, and sterile neutrinos produced non-resonantly via the (DW) mechanism[32] for which the chemical potential vanishes and $I_F[0] = 2 \ln(2)/3\zeta(3) = 0.3844$, and the label s refers solely to sterile neutrinos produced via a lepton-driven (MSW) resonance via the mechanism described in ref.[33]. The label B corresponds to condensed or non-condensed Bosons of mass m and chemical potential μ that decoupled at temperature T_d while ultrarelativistic. The function I_B features an infrared divergence in the limits $\mu/T_d; m/T_d \rightarrow 0$ or $\mu = m$ for any value of the mass. This latter case corresponds to the case of a Bose-Einstein Condensate[34]. Thus Bosonic particles that decoupled while ultrarelativistic with

$\mu/T_d \ll 1$ or that formed a BEC lead to small free-streaming lengths and behave as CDM. Finally, (WIMPs) are considered to be decoupled while non-relativistic with a Maxwell-Boltzmann distribution function.

Eq.(1.2) clearly shows that (WIMPs) with $m \sim 100$ GeV that decoupled kinetically in LTE at $T \sim 10$ MeV[35] dominate all other contributions to k_{fs} resulting in an extremely small free streaming length $\lambda_{fs} \sim 6.5 \times 10^{-3}$ pc unless their fractional abundance $\nu_{wimp} \lesssim 10^{-12}$.

If (DM) is dominated by sterile neutrinos (produced either resonantly or non-resonantly) that decoupled near the QCD scale[32, 33] with $m \sim$ keV, we find that the typical free-streaming lengths *today* are $\lambda_{fs} \sim 2 - 7$ kpc, where the larger value corresponds to the non-resonant and the lower value to the resonant production mechanisms respectively. The lower values are consistent with the recent numerical study of the formation of the first stars in filamentary structures[19] and the values for the “cores” extracted from the data in ref.[14] for dwarf spheroidal galaxies (dSphs) ($r_c \sim 0.5$ kpc). The upper values agree with the cores extracted from the data in ref.[15] for spiral galaxies ($r_c \sim 10$ kpc).

II. FREE STREAMING LENGTH FOR MULTICOMPONENT DARK MATTER

We study DM particles that decoupled in or out of equilibrium while relativistic or non-relativistic, but that are non-relativistic *today*. As argued above, Gilbert’s detailed numerical study[25] confirmed by Bond and Szalay[26] shows that the value of k_{fs} can be extracted from the marginal case between modes that grow under gravitational instability and those that are Landau damped. For this marginal case linear perturbations are stationary, and just as in the case of the Jeans length this marginal value can be reliably extracted in a non-expanding cosmology, including *a posteriori* the scale factor dependence of the various quantities in the Jeans wavelength which separates the gravitationally stable and unstable modes[18]. In section (III A) we present arguments that determine the redshift dependence of the free streaming length under a suitable approximation.

Consider several species of DM candidates that are non-relativistic *today* with masses m_a and distribution functions f_a where the label $a = 1, 2, \dots$ refers to the different components. Each component (a) obeys the collisionless Boltzmann-Vlasov equation[20, 21, 23],

$$\frac{\partial f_a(\vec{x}, \vec{p}; t)}{\partial t} + \frac{\vec{p}}{m_a} \cdot \vec{\nabla}_{\vec{x}} f_a(\vec{x}, \vec{p}; t) - m_a \vec{\nabla}_{\vec{x}} \Phi(\vec{x}; t) \vec{\nabla}_{\vec{p}} f_a(\vec{x}, \vec{p}; t) = 0 \quad (2.1)$$

where Φ is the total Newtonian potential which is the solution of the Poisson equation

$$\nabla^2 \Phi(\vec{x}; t) = 4\pi G \sum_a m_a g_a \int \frac{d^3 p}{(2\pi)^3} f_a(\vec{x}, \vec{p}; t), \quad (2.2)$$

where g_a is the number of internal degrees of freedom.

We note that whereas each individual species obeys its own collisionless Boltzmann-Vlasov equation, the Newtonian gravitational potential is determined by *all the components* as indicated by the Poisson equation (2.2). Therefore, although the different DM components do not interact *directly* they interact *indirectly* via the self-consistent gravitational potential since *all* of the DM components act as source of this potential which enters in Boltzmann-Vlasov equation of each component.

Linearizing the Boltzmann-Vlasov equation and writing

$$f_a(\vec{x}, \vec{p}; t) = f_a^{(0)}(p) + f_a^{(1)}(\vec{x}, \vec{p}; t) \quad ; \quad \Phi(\vec{x}; t) = \Phi^0(\vec{x}; t) + \Phi^{(1)}(\vec{x}; t) \quad (2.3)$$

where $f_a^{(0)}(p)$ are the distribution functions of the species that *decoupled in or out of LTE*, the *only assumption* is that these are isotropic, namely only depend on $p = |\vec{p}|$. In the non-expanding case the zeroth order equation requires to invoke the usual “Jeans swindle” (see the textbooks[23]) whereas in the expanding case the zeroth order equation is solved in terms of the inhomogeneous gravitational potential which yields the expanding background (see[20]).

It is convenient to perform a spatial Fourier transform of the perturbations in a volume V

$$f_a^{(1)}(\vec{x}, \vec{p}; t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} F_a^{(1)}(\vec{k}, \vec{p}; t) e^{i\vec{k} \cdot \vec{x}} \quad ; \quad \Phi^{(1)}(\vec{x}; t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{x}} \varphi^{(1)}(\vec{k}; t) \quad (2.4)$$

in terms of which the Vlasov and Poisson equations for the perturbations become

$$\frac{dF_a^{(1)}(\vec{k}, \vec{p}; t)}{dt} + i \frac{\vec{k} \cdot \vec{p}}{m_a} F_a^{(1)}(\vec{k}, \vec{p}; t) - i m_a \vec{k} \cdot \hat{\vec{p}} \varphi_1(\vec{k}; t) \frac{df_a^{(0)}(p)}{dp} = 0, \quad (2.5)$$

$$\varphi_1(\vec{k}; t) = -\frac{4\pi G}{k^2} \sum_b m_b g_b \int \frac{d^3 p}{(2\pi)^3} F_b^{(1)}(\vec{k}, \vec{p}; t). \quad (2.6)$$

Inserting eqn. (2.6) into eqn. (2.5) makes explicit that all the DM components are actually *interacting* through the self-consistent gravitational perturbations which act as a dynamical self-consistent mean field, as discussed above.

Decaying or growing perturbations must be treated as an initial value problem, and following the treatment of Landau damping in plasmas[22] we introduce the Laplace transform of the perturbations

$$\tilde{F}_a^{(1)}(\vec{k}, \vec{p}; s) = \int_0^\infty e^{-st} F_a^{(1)}(\vec{k}, \vec{p}; t) dt \quad ; \quad \tilde{\varphi}^{(1)}(\vec{k}; s) = \int_0^\infty e^{-st} \varphi^{(1)}(\vec{k}; t) dt. \quad (2.7)$$

The Laplace transform of the Boltzmann-Vlasov equation (2.5) leads to

$$\tilde{F}_a^{(1)}(\vec{k}, \vec{p}; s) = \tilde{\varphi}^{(1)}(\vec{k}; s) \frac{im_a \vec{k} \cdot \hat{\vec{p}}}{s + i \frac{\vec{k} \cdot \vec{p}}{m_a}} \frac{df_a^{(0)}(p)}{dp} + \frac{F_a^{(1)}(\vec{k}, \vec{p}; t=0)}{s + i \frac{\vec{k} \cdot \vec{p}}{m_a}}. \quad (2.8)$$

Taking the Laplace transform of (2.6), multiplying (2.8) by $(-4\pi G m_a g_a / k^2)$, summing over a and integrating in p we are led to

$$\tilde{\varphi}^{(1)}(\vec{k}; s) = \frac{i 4\pi G}{k^2 \varepsilon(k; s)} \sum_a m_a g_a \int \frac{d^3 p}{(2\pi)^3} \frac{F_a^{(1)}(\vec{k}, \vec{p}; t=0)}{\frac{\vec{k} \cdot \vec{p}}{m_a} - is} \quad (2.9)$$

where the gravitational “polarization” function is given by

$$\varepsilon(k; s) = 1 + \frac{4\pi G}{k^2} \sum_a m_a^2 g_a \int \frac{d^3 p}{(2\pi)^3} \frac{\vec{k} \cdot \hat{\vec{p}}}{\frac{\vec{k} \cdot \vec{p}}{m_a} - is} \frac{df_a^{(0)}(p)}{dp} \quad (2.10)$$

The collective excitations of the collisionless self-gravitating system correspond to the poles of $\tilde{\varphi}^{(1)}(\vec{k}; s)$ in the complex s -plane, these are the zeroes of $\varepsilon(k; s)$. The time dependence of the perturbation of the Newtonian gravitational potential is obtained by the inverse Laplace transform

$$\varphi^{(1)}(\vec{k}; t) = \int_C \frac{ds}{2\pi i} e^{st} \tilde{\varphi}^{(1)}(\vec{k}; s) \quad (2.11)$$

where C stands for the Bromwich contour parallel to the imaginary axis and to the right of all the singularities of $\tilde{\varphi}^{(1)}(\vec{k}; s)$ in the complex s -plane.

Using that $f_a^{(0)}(p)$ is a function of $p = |\vec{p}|$ it is convenient to write

$$\frac{\vec{k} \cdot \hat{\vec{p}}}{\frac{\vec{k} \cdot \vec{p}}{m_a} - is} = \frac{m_a}{p} \left[1 + \frac{is}{\frac{\vec{k} \cdot \vec{p}}{m_a} - is} \right] \quad (2.12)$$

which allows to extract the $s = 0$ contribution. The resulting expression for $\varepsilon(k; s)$ becomes

$$\varepsilon(k; s) = \varepsilon(k; 0) + is \frac{4\pi G}{k^2} \sum_a g_a m_a^3 \int \frac{d^3 p}{(2\pi)^3} \frac{\frac{df_a^{(0)}(p)}{p dp}}{\frac{\vec{k} \cdot \vec{p}}{m_a} - is} \quad (2.13)$$

where

$$\varepsilon(k; 0) = 1 + \frac{4\pi G}{k^2} \sum_a g_a m_a^3 \int_0^\infty p \frac{df_a^{(0)}(p)}{dp} \frac{dp}{2\pi^2} \equiv 1 - \frac{k_{fs}^2}{k^2}. \quad (2.14)$$

In eqn. (2.14) we have integrated by parts and introduced the *free streaming* momentum k_{fs} given by

$$k_{fs}^2 = 4\pi G \sum_a \rho_a^{(0)} \left\langle \frac{1}{V^2} \right\rangle_a \quad (2.15)$$

where

$$\rho_a^{(0)} = m_a n_a^{(0)} \quad ; \quad n_a^{(0)} = g_a \int \frac{d^3 p}{(2\pi)^3} f_a^{(0)}(p) \quad (2.16)$$

and

$$\left\langle \frac{1}{\vec{V}^2} \right\rangle_a = \frac{g_a}{n_a^{(0)}} \int \frac{d^3 p}{(2\pi)^3} \frac{m_a^2}{p^2} f_a^{(0)}(p), \quad (2.17)$$

the free streaming length is obtained as

$$\lambda_{fs} = \frac{2\pi}{k_{fs}}. \quad (2.18)$$

The expression for k_{fs} can be written as

$$k_{fs}^2 = \frac{\Omega_{DM} h^2}{(\text{Mpc})^2} \sum_a \nu_a \left\langle \frac{V_0^2}{\vec{V}^2} \right\rangle_a \quad ; \quad V_0 = 122.5 \frac{\text{km}}{\text{s}} \quad (2.19)$$

where the *partial fractions* are defined as

$$\nu_a = \frac{\Omega_a}{\Omega_{DM}} \quad ; \quad \sum_a \nu_a = 1. \quad (2.20)$$

The *collective* modes correspond to the *zeroes* of the “gravitational polarization” function $\varepsilon(k; s)$ eqn. (2.13) in the complex s - plane. These yield the time evolution for the gravitational perturbations

$$\varphi^{(1)}(\vec{k}; t) \propto \sum_p e^{s_p(k)t} \quad (2.21)$$

where $s_p(k)$ are the zeroes of $\varepsilon(k; s)$.

It is illuminating to compare the expression (2.15) with that for the Jeans wavevector for a single fluid

$$k_J^2 = \frac{4\pi G \rho^{(0)}}{c_s^2}, \quad (2.22)$$

where c_s is the adiabatic speed of sound of the fluid. We see that for a single collisionless component we can obtain the free streaming length from the Jeans length by the replacement

$$c_s \Rightarrow \left[\left\langle \frac{1}{\vec{V}^2} \right\rangle \right]^{-\frac{1}{2}} \quad (2.23)$$

which in general *is different* from replacing c_s by the velocity dispersion $\sqrt{\langle \vec{V}^2 \rangle}$. This difference becomes important when the unperturbed distribution function favors small values of the momentum. This observation will become crucial when we study Bosonic particles and sterile neutrinos decoupled when relativistic but *out of LTE*. As it will be seen in detail below, in these cases the distribution function favors the region of small momentum which leads to dramatic consequences in the difference between $1/\langle \vec{V}^2 \rangle$ and $\langle 1/\vec{V}^2 \rangle$.

A. Landau damping and Jeans instability

It is clear from (2.13) that there is a pole in the Laplace transform of the Newtonian perturbation for the marginal value

$$s = 0 \quad ; \quad k = k_{fs}. \quad (2.24)$$

This is akin to the marginal value $k = k_J$ in a fluid where k_J is the Jeans wave vector, in a fluid for $k > k_J$ pressure gradients hinder gravitational collapse and the perturbations are simple acoustic oscillations, for $k < k_J$ pressure gradients cannot prevent the collapse and the self-gravitational fluid undergoes the Jeans instability towards

gravitational collapse. We can study the dynamics of collective excitations in region $k \approx k_{fs}$ searching for zeroes in $\varepsilon(k; s)$ for $s \approx 0$. The second term in (2.13) can be evaluated by performing the angular integral and using that $f_a^{(0)}$ only depends on $p = |\vec{p}|$, namely

$$\int_{-1}^1 \frac{d(\cos \theta)}{\frac{kp \cos \theta}{m_a} - is} = \frac{m_a}{kp} \ln \left[\frac{\frac{kp}{m_a} - is}{-\frac{kp}{m_a} - is} \right]. \quad (2.25)$$

For $s \approx 0$ the branch is defined by the prescription determined by the Bromwich contour $Re(s) > 0$, which is precisely recognized as the Landau prescription for the evaluation of the integrals[22]. We find

$$\varepsilon(k; s) = 1 - \frac{k_{fs}^2}{k^2} + is \frac{G}{\pi k^3} \sum_a g_a m_a^4 \int_0^\infty \left[-\frac{df_a^{(0)}(p)}{dp} \right] \left\{ \ln \left[\frac{\frac{kp}{m_a} + is}{-\frac{kp}{m_a} - is} \right] - i\pi \right\} dp \quad (2.26)$$

Because for small s the logarithm in (2.26) is linear in s , the last term in the bracket ($-i\pi$) contributes to the leading order in $k^2 - k_{fs}^2$. To lowest order in s for $k \approx k_{fs}$, the condition $\varepsilon(k; s) = 0$ yields

$$s(k) = \mathcal{C} [k_{fs}^2 - k^2] \quad ; \quad \mathcal{C} = \frac{k}{G \sum_a g_a m_a^4 f_a^{(0)}(0)} > 0. \quad (2.27)$$

From the time evolution of the gravitational perturbation eqn. (2.21) we find

$$\begin{aligned} s(k) < 0 \text{ for } k > k_{fs} &\Rightarrow \text{Landau damping} \\ s(k) > 0 \text{ for } k < k_{fs} &\Rightarrow \text{Jeans instability.} \end{aligned} \quad (2.28)$$

The long wavelength limit $k \rightarrow 0$ is obtained by expanding $(\frac{\vec{k} \cdot \vec{p}}{m} - is)^{-1}$ in the integrand in eqn. (2.13) in powers of $\vec{k} \cdot \vec{p}/ms$. In the resulting expression only the odd powers survive the angular integration. Keeping up to $(\vec{k} \cdot \vec{p}/ms)^3$ the long wavelength limit of $\varepsilon(k; s)$ is found to be

$$\varepsilon(k; s) = 1 - \frac{4\pi G}{s^2} \sum_a \rho_a^{(0)} \left[1 - \langle V^2 \rangle_a \frac{k^2}{s^2} + \dots \right] \quad (2.29)$$

and the zeroes of $\varepsilon(k; s)$ in the long-wavelength limit are found to be

$$s_{\pm}(k) = \pm \left[\Omega_J^2 - \overline{V^2} k^2 \right]^{\frac{1}{2}} + \dots \quad (2.30)$$

where

$$\Omega_J^2 = 4\pi G \sum_a \rho_a^{(0)} \quad ; \quad \overline{V^2} = \sum_a \nu_a \langle \vec{V}^2 \rangle_a \quad (2.31)$$

and ν_a are the partial fractions. The Jeans frequency Ω_J is the same as that for single component fluids, however the relationship between the free streaming wavevector k_{fs} (2.15) and the Jeans frequency Ω_J is different from that of the Jeans wavevector and the Jeans frequency in a single component fluid. For a single collisionless component we find

$$\Omega_J = \left[\left\langle \frac{1}{\vec{V}^2} \right\rangle \right]^{-\frac{1}{2}} k_{fs} \quad (2.32)$$

whereas for a single *fluid* one finds[20]

$$\Omega_J = c_s k_J. \quad (2.33)$$

B. An example: the Maxwell-Boltzmann distribution

In general the momentum integral in (2.10) cannot be found in closed form without approximating the distribution function. However the Maxwell-Boltzmann distribution provides an example for which a closed form expression for

(2.10) can be found. This distribution function is relevant for the description of WIMPs which are heavy relics that decoupled in LTE while non-relativistic[18]. In this case

$$f^{(0)}(\vec{p}) = \mathcal{N} e^{-\frac{\vec{p}^2}{2mT_d}} \quad (2.34)$$

where the normalization \mathcal{N} is obtained from the solution of the kinetic equation for the distribution function, which can be found in section (5.2) in ref.[18]. The particle density is

$$\rho = mg \int \frac{d^3p}{(2\pi)^3} f^{(0)}(\vec{p}) = \mathcal{N} mg \left[\frac{mT_d}{2\pi} \right]^{\frac{3}{2}}. \quad (2.35)$$

The momentum integrals in eqn. (2.10) can be carried out straightforwardly because $f^{(0)}(\vec{p})$ is a function of \vec{p}^2 . This is achieved by splitting the vector \vec{p} into components parallel and perpendicular to \vec{k} . With $d^3p = dp_{\parallel} d^2p_{\perp}$ the integrals along the parallel and perpendicular directions can be done straightforwardly. We obtain the result

$$\varepsilon(k; s) = 1 - \frac{k_{fs}^2}{k^2} + \frac{s}{k^3} g \mathcal{N} m^4 e^{\delta^2} \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\delta} e^{-t^2} dt \right] ; \quad \delta = \frac{s}{k} \left[\frac{m}{2T_d} \right]^{\frac{1}{2}}. \quad (2.36)$$

It is straightforward to analytically continue this function to $\text{Re}(s) < 0$ according to the Landau prescription[22].

With $f^{(0)}(\vec{0}) = \mathcal{N}$ we find for the marginal case $s = 0$ the solution $k^2 = k_{fs}^2$, where

$$k_{fs}^2 = 4\pi G \rho \left\langle \frac{1}{\vec{V}^2} \right\rangle, \quad (2.37)$$

and ρ is given by eqn. (2.35). Furthermore it is a simple exercise to confirm that near the marginal case the pole in the function $\varepsilon(k; s)$ is given by eqn. (2.27). For the Maxwell-Boltzmann distribution it follows that

$$\left\langle \vec{V}^2 \right\rangle = \frac{3T}{m} ; \quad \left\langle \frac{1}{\vec{V}^2} \right\rangle = \frac{m}{T} \neq \frac{1}{\left\langle \vec{V}^2 \right\rangle}. \quad (2.38)$$

The long-wavelength limit can be obtained by expanding the integral in (2.36) for $\delta \gg 1$, namely

$$\left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\delta} e^{-t^2} dt \right] = \frac{e^{-\delta^2}}{\delta \sqrt{\pi}} \left[1 - \frac{1}{2\delta^2} + \frac{3}{4\delta^4} + \dots \right] \quad (2.39)$$

which after tedious but straightforward algebra leads to the expressions for the poles given by eqn. (2.31) with $\nu = 1$ and $\Omega_J^2 = 4\pi G \rho$ where ρ is given by (2.35) and $\left\langle \vec{V}^2 \right\rangle$ given by (2.38).

III. FREE STREAMING LENGTHS FOR DECOUPLED PARTICLES

The distribution function of decoupled particles in a homogeneous and isotropic cosmological background in absence of gravitational perturbations is constant along geodesics and obey the Liouville or collisionless Boltzmann equation in terms of an affine parameter λ [18, 34, 36]

$$\frac{d}{d\lambda} f[P_f; t] = 0 \Rightarrow \frac{df[P_f; t]}{dt} = 0 \quad (3.1)$$

where $P_f = p_c/a(t)$ is the physical momentum, and p_c the time independent comoving momentum. Taking P_f as an independent variable this equation leads to the familiar form[18, 34, 36]

$$\frac{\partial f[P_f; t]}{\partial t} - H P_f \frac{\partial f[P_f; t]}{\partial P_f} = 0, \quad (3.2)$$

where $H = \dot{a}/a$ is the Hubble parameter and $P_f a = p_c = \text{constant}$ is a characteristic of the equation. Obviously a solution of this equation is

$$f[P_f; t] \equiv f_d[a(t)P_f] = f_d[p_c]. \quad (3.3)$$

If a particle of mass m has been in LTE but it decoupled from the plasma with decoupling temperature T_d its distribution function is

$$f_d(p_c) = \frac{1}{e^{\frac{\sqrt{m^2 + p_c^2} - \mu_d}{T_d}} \pm 1} \quad (3.4)$$

for Fermions (+) or Bosons (−) respectively allowing for a chemical potential.

Since the distribution function is dimensionless, without loss of generality we can always write for a *decoupled* particle[34]

$$f_d(p_c) = f_d\left(\frac{p_c}{T_d}; \frac{m}{T_d}; \alpha_i\right) \quad (3.5)$$

where α_i are a collection of *dimensionless* constants determined by the microphysics, for example dimensionless couplings or ratios between T_d and particle physics scales or in equilibrium μ_d/T_d etc. To simplify notation in what follows we will not include explicitly the set of dimensionless constants $m/T_d; \alpha_i$, etc, in the argument of f_d , but these are implicit in generic distribution functions. If the particle decouples when it is still relativistic $m/T_d \rightarrow 0$.

It is convenient to introduce the dimensionless ratios[34]

$$y = \frac{p_c}{T_d} = \frac{P_f}{T_d(t)} \quad ; \quad T_d(t) = \frac{T_d}{a(t)} \quad ; \quad x_d = \frac{m}{T_d}. \quad (3.6)$$

We emphasize that the distribution functions (3.5) are general and *not* necessarily describing particles decoupled while in local thermal equilibrium. When the particle becomes non-relativistic, its contribution to the energy density is

$$\rho = m n(t) \quad (3.7)$$

where[34]

$$n(t) = g \frac{T_d^3(t)}{2\pi^2} \int_0^\infty y^2 f_d(y) dy \quad (3.8)$$

and g is the number of internal degrees of freedom. From entropy conservation[18, 36], the decoupling temperature at redshift z is related to the temperature of the CMB today by

$$T_d(z) = (1+z) \left(\frac{2}{g_d}\right)^{\frac{1}{3}} T_{cmb} = \left(\frac{2}{g_d}\right)^{\frac{1}{3}} 2.348 \times 10^{-4} (1+z) \text{ eV} , \quad (3.9)$$

where g_d is the number of effective ultrarelativistic degrees of freedom at decoupling. For a given species (a) of particles with g_a internal degrees of freedom that decouples when the effective number of ultrarelativistic degrees of freedom is $g_{d,a}$, the relic abundance *today* ($z = 0$) is given by[34]

$$\Omega_a h^2 = \left(\frac{m_a}{25.67 \text{ eV}}\right) \frac{g_a \int_0^\infty y^2 f_{d,a}(y) dy}{2g_{d,a} \zeta(3)}. \quad (3.10)$$

If this decoupled species contributes a fraction ν_a to dark matter, with $\Omega_a = \nu_a \Omega_{DM}$ and taking $\Omega_{DM} h^2 = 0.105$ [37] for non-baryonic dark matter, we find[34]

$$\nu_a = \left(\frac{m_a}{6.227 \text{ eV}}\right) \frac{g_a}{g_{d,a}} \int_0^\infty y^2 f_{d,a}(y) dy. \quad (3.11)$$

The constraint on the fractional abundance $0 \leq \nu_a \leq 1$ yields a bound for the mass of the particle[34].

We are now in position to establish a relation between the distribution function of the decoupled particles as dark matter candidates and the results from the Boltzmann-Vlasov equation which determine the free streaming length.

The analysis of the Boltzmann-Vlasov equation in the non-expanding case applies to the description of perturbations at very low redshift in the Newtonian approximation. Such approximation is correct provided the relevant length scales, for example the free streaming length, are much smaller than the Hubble radius. Such is the case whenever

the velocity dispersion of the particles is $(V/c)^2 \ll 1$. The momentum that enters in the Boltzmann-Vlasov equation is the physical momentum, which in the non-relativistic limit is related to the velocity as

$$\vec{V}^2 = \frac{\vec{P}_f^2}{m^2}. \quad (3.12)$$

The unperturbed distribution functions for decoupled particles that enter in the linearized Boltzmann-Vlasov eqn. (2.5) are the solutions of the unperturbed collisionless Boltzmann equation, namely

$$f^{(0)}(p) = f_d^{(0)}(p_c) \quad (3.13)$$

where $f_d(p_c)$ are given by eqn. (3.5). Restoring the speed of light c , the average

$$\left\langle \frac{1}{\vec{V}^2} \right\rangle_a = \left(\frac{m_a}{T_{d,a}(z)c} \right)^2 I_a, \quad (3.14)$$

where $T_{d,a}(z)$ is given by eqn. (3.9) for the species a , and we have introduced

$$I_a \equiv \left[\frac{\int_0^\infty f_{d,a}^{(0)}(y) dy}{\int_0^\infty y^2 f_{d,a}^{(0)}(y) dy} \right]. \quad (3.15)$$

This dimensionless ratio of integrals only depends on the ratios $m_a/T_d, \mu_a/T_d$ and dimensionless couplings from the microphysics at decoupling. Using eqn. (3.9) we obtain

$$\left\langle \frac{V_0^2}{\vec{V}^2} \right\rangle_a = \frac{1.905}{(1+z)^2} \left(\frac{m_a}{\text{eV}} \right)^2 g_{d,a}^{\frac{2}{3}} I_a. \quad (3.16)$$

Inserting this result into the expression (2.19) and using $\Omega_{DM} h^2 = 0.105$ [37] for non-baryonic dark matter we find *today*, for $z = 0$

$$k_{fs}^2 = \left[\frac{0.447}{\text{kpc}} \right]^2 \sum_a \nu_a g_{d,a}^{\frac{2}{3}} \left(\frac{m_a}{\text{keV}} \right)^2 I_a. \quad (3.17)$$

For light particles that decouple while they are ultrarelativistic the distribution function $f_{d,a}(y)$ does not depend on the ratio m_a/T_d , however, for particles that decouple when they are non-relativistic, their distribution function is typically a Maxwell-Boltzmann distribution which does depend on this ratio.

Inserting the result (3.11) for the fractions (3.11) and (3.16) into eqn. (3.17) leads to the alternative form

$$k_{fs}^2 = \left[\frac{5.632}{\text{kpc}} \right]^2 \sum_a \frac{g_a}{g_{d,a}^{\frac{1}{3}}} \left(\frac{m_a}{\text{keV}} \right)^3 \int_0^\infty f_{d,a}^{(0)}(y) dy \quad (3.18)$$

where g_a are the internal degrees of freedom of the particle of species (a). The simple expressions (3.17,3.18) are some of the main results of this article.

A. Redshift dependence

The Boltzmann-Vlasov equation in a non-expanding cosmology (2.1) is obtained by setting the scale factor $a \equiv 1$ in the same equation in an expanding cosmology (see ref.[20, 21]). Thus the linearized equation describes the properties of the collective excitations *today*. In an expanding cosmology the decay or growth of perturbations is no longer exponential but typically a power of the scale factor[20, 21]. However, here we are *not* concerned directly with the manner in which linear perturbations grow or decay, but with the marginal wave-vector k_{fs} that determines the crossover of behavior from growth to damping of collective excitations. If we *assume* that the expansion is sufficiently slow that it can be treated adiabatically we can obtain the redshift behavior of the free-streaming length $\lambda_{fs}(z)$ by replacing the densities, velocities and wavevectors with the corresponding scale factors given by eqns. (3.8,3.9,3.16), namely:

$$\begin{aligned} \rho &\rightarrow \rho(z) = \rho(0)(1+z)^3 \\ \left\langle \frac{1}{\vec{V}^2} \right\rangle &\rightarrow \left\langle \frac{1}{\vec{V}^2(z)} \right\rangle = \left\langle \frac{1}{\vec{V}^2(0)} \right\rangle (1+z)^{-2} \\ k &\rightarrow k(1+z), \end{aligned} \quad (3.19)$$

where $z = 0$ refers to *today*. Defining the *comoving* free-streaming wavevector $k_{fs}(z) = 2\pi/\lambda_{fs}(z)$ upon rescaling by the corresponding scale factor

$$k_{fs} \rightarrow k_{fs}(z)(1+z). \quad (3.20)$$

Assuming the validity of this adiabatic scaling in eqn (2.15) we obtain,

$$k_{fs}^2(z) = \frac{4\pi G}{(1+z)^2} \sum_a \rho^{(0)}(z) \left\langle \frac{1}{\vec{V}^2(z)} \right\rangle. \quad (3.21)$$

Since the DM density and velocity dispersions for all components scale as in eqn. (3.19) we find the following redshift dependence of the *comoving* free-streaming wavevector

$$k_{fs}^2(z) = \frac{4\pi G}{(1+z)} \sum_a \rho^{(0)}(0) \left\langle \frac{1}{\vec{V}^2(0)} \right\rangle. \quad (3.22)$$

This result is similar to the expression for the Jeans's wavevector in a Newtonian fluid[20, 21] upon replacing $\langle 1/\vec{V}^2(z) \rangle \rightarrow c_s^2(z)$ where $c_s(z)$ is the (adiabatic) speed of sound in the medium as a function of redshift. In fact the validity of this assumption is confirmed not only by the similarity with the familiar Jeans' result for Newtonian fluids in an expanding cosmology, but also by the exact solution obtained in ref.[29] for the case of neutrinos decoupled in LTE. Therefore we identify k_{fs} given by eqn. (3.17) as the *comoving* free streaming wave-vector. The scaling behavior of the *comoving* free streaming length $\lambda_{fs}(z) = \lambda_{fs}(0)\sqrt{(1+z)}$ leads to the free-streaming mass

$$M_{fs}(z) = \frac{4\pi}{3} \sum_a \rho_a^{(0)}(z) \left(\frac{\lambda_{fs}(z)}{1+z} \right)^3 = M_{fs}(0)(1+z)^{\frac{3}{2}}, \quad (3.23)$$

a relation similar to the that of the Jeans' mass in the non-relativistic regime. Therefore, under the validity of the adiabatic assumption, the simple re-scaling of the free-streaming wave-vector and length given by eqn. (3.22) indicates that we can simply obtain these quantities *today* ($z = 0$) and extrapolate to an arbitrary redshift z via eqn. (3.22) provided the redshift is still small enough that the species are non-relativistic.

The validity of the adiabatic assumption relies on the fact that in the non-relativistic regime with $\langle \vec{V}^2/c^2 \rangle \ll 1$, the free-streaming length is *much smaller* than the Hubble radius, which is found below to be a consistent assumption, or alternatively $k_{fs}/H \gg 1$. And as mentioned above the result for the free streaming length obtained from this adiabatic hypothesis is similar to the usual result for the Jeans' length[20, 21] and is confirmed in ref.[29] for the case of a neutrino thermal relic.

A more detailed analysis of Gilbert's equation for *mixtures* of DM components with arbitrary (but isotropic) distribution functions in the adiabatic approximation will be provided elsewhere[38].

We now gather the above results to give the general expression for the free streaming length of an arbitrary *mixture* of non-relativistic species that decoupled in or out of LTE either ultrarelativistic or non-relativistic, in terms of the *partial fraction* (ν) that each contributes to the (DM) content and the dimensionless ratios I_a

$$\frac{1}{\lambda_{fs}^2(z)} = \frac{1}{(1+z)} \left[\frac{0.071}{\text{kpc}} \right]^2 \sum_a \nu_a g_{d,a}^{\frac{2}{3}} \left(\frac{m_a}{\text{keV}} \right)^2 I_a, \quad (3.24)$$

where

$$I_a = \left[\frac{\int_0^\infty f_{d,a}^{(0)}(y) dy}{\int_0^\infty y^2 f_{d,a}^{(0)}(y) dy} \right] \quad (3.25)$$

in terms of the general distribution functions (3.5) which only depend on the ratios $m_a/T_d, \mu_a/T_d$ and dimensionless couplings and are completely determined by the microphysics at decoupling.

We now proceed to obtain the contributions to the free streaming wave-vector *today* ($z = 0$) from the various components: thermal relics that decoupled either relativistic or non-relativistic in LTE and non-thermal relics that decoupled while relativistic but out of LTE.

B. Thermal relics

Let us consider ultrarelativistic Fermionic or Bosonic particles decoupled in LTE with chemical potentials and with $m_a/T_{d,a} \ll 1$.

- **Ultrarelativistic Fermions:** Neglecting m/T_d in the ultrarelativistic limit, but keeping the chemical potential μ , the distribution function is

$$f_d^{(0)}(y) = \frac{1}{e^{(y-u)} + 1} \quad ; \quad u = \frac{\mu}{T_d} \quad (3.26)$$

where we have neglected $m/T_d \ll 1$. For this distribution

$$\int_0^\infty f_d^{(0)}(y) dy = \ln[1 + e^u] \quad (3.27)$$

Combining this result with eqn. (3.18) we note that larger chemical potentials lead to *shorter* free streaming scales.

Denote $I_F[u]$ the ratio I_a , eqn. (3.15) for an ultrarelativistic Fermionic thermal relic with chemical potential μ . It is depicted in fig. (1) as a function of $u = \mu/T_d$. For $\mu = 0$ we find

$$I_F[0] = \frac{2 \ln(2)}{3 \zeta(3)} = 0.3844. \quad (3.28)$$

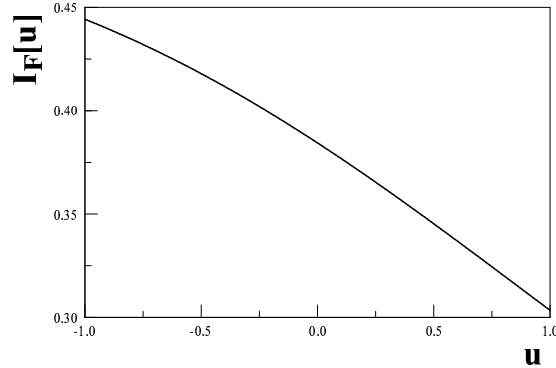


FIG. 1: $I_F[u]$ vs $u = \mu/T_d$ for $f_d(y) = 1/(e^{(y-u)} + 1)$. $I_F[0] = 0.3844$.

For $\mu = 0$ and taking $\nu = 1$ for this Fermionic species and vanishing abundance for all others, the result for k_{fs} given by eqn. (3.17) with $I_a = 2 \ln(2)/3 \zeta(3)$ agrees with that found in ref.[29]. Therefore, a Fermionic thermal relic as a unique DM component yields a free-streaming length *today* ($z = 0$)

$$\lambda_{fs}(0) = \frac{14}{g_d^{\frac{1}{3}} \sqrt{I_F[u]}} \left(\frac{\text{keV}}{m} \right) \text{kpc}. \quad (3.29)$$

For example a neutrino with $m \sim \text{eV}$ decoupling at $T_d \sim 1 \text{ MeV}$, when $g_d \sim 10$, yields a free streaming length today

$$\lambda_{fs}(0) \sim 10 \text{ Mpc}, \quad (3.30)$$

which is the usual estimate for free-streaming lengths for HDM candidates.

- **Ultrarelativistic Bosons:** The distribution function is

$$f_d^{(0)}(y) = \frac{1}{e^{(y-u_d)} - 1} \quad ; \quad u_d = \frac{\mu}{T_d} \quad (3.31)$$

where we have neglected $m/T_d \ll 1$ and in this case the chemical potential $\mu \leq 0$ so that the distribution function (a probability) is manifestly positive definite. For this distribution

$$\int_0^\infty f_d^{(0)}(y) dy = -\ln[1 - e^{-|u_d|}] \quad (3.32)$$

This expression clearly reveals that for $m = 0, \mu_d = 0$ the distribution function diverges logarithmically at $p = 0$. In an expanding cosmology with a particle horizon the smallest wavevector that can describe causal microscopic physics is determined by the Hubble scale. Since we are considering decoupling in LTE, it is consistent to assume an infrared momentum cutoff of order $H_d \sim \sqrt{g_d} T_d^2 / M_{Pl}$, the Hubble parameter at the time of decoupling. Therefore *assuming* such an infrared cutoff, keeping $m/T_d \ll 1$ but non-zero and with vanishing chemical potential we find

$$\int_0^\infty f_d^{(0)}(y) dy \sim \ln \left[\frac{2 T_d}{H_d + \sqrt{H_d^2 + m^2}} \right]. \quad (3.33)$$

In the denominator in eqn. (3.15) we can set $x_d = 0$ in the ultrarelativistic limit and for vanishing chemical potential we find

$$I_B[x_d, 0] \sim \frac{1}{2\zeta(3)} \ln \left[\frac{2 T_d}{H_d + \sqrt{H_d^2 + m^2}} \right]. \quad (3.34)$$

Keeping the mass and the chemical potential the distribution function becomes

$$f_d^{(0)}(y) = \frac{1}{e^{\sqrt{y^2 + x_d^2} - u_d} - 1} \quad ; \quad x_d = \frac{m}{T_d} \quad ; \quad u_d = \frac{\mu}{T_d} \quad (3.35)$$

for which we find

$$\int_0^\infty f_d^{(0)}(y) dy = x_d \sum_{l=1}^\infty e^{-l|u_d|} K_1[l x_d], \quad (3.36)$$

Where K_1 is a Hankel function. For $m/T_d, \mu/T_d \ll 1$ we can neglect the mass and chemical potential in the denominator in eqn. (3.15) and we find for an ultrarelativistic boson with a non-vanishing chemical potential in the limit $m/T_d, \mu/T_d \ll 1$

$$I_B[x_d; u_d] \simeq \frac{x_d}{2\zeta(3)} \sum_{l=1}^\infty e^{-l|u_d|} K_1[l x_d]. \quad (3.37)$$

This function features an infrared divergence in the limit $m/T_d, \mu/T_d \rightarrow 0$ from the numerator in (3.15) given by (3.36), which must be regulated by assuming an infrared cutoff of the order of the Hubble scale at decoupling as in eqn. (3.33). A more thorough analysis of the infrared behavior of the decoupled distribution function in an expanding cosmology is necessary, in particular an assessment of the (causal) thermalization process for superhorizon wavelengths. Such program, interesting all by itself, is clearly beyond the realm of this study.

Bose-Einstein Condensation: If chemical freeze out occurs before kinetic decoupling, it is possible that bosonic particles undergo Bose-Einstein condensation[34]. Under this circumstance the homogeneous condensate ($\vec{p} = 0$) contributes to the dynamics of the scale factor and inhomogeneous perturbations only modify the distribution function of the particles outside of the condensate. The linearized Boltzmann-Vlasov equation therefore applies to the non-condensate part, since the dynamics of the condensate cannot be described by the incoherent Boltzmann equation but by the equation of motion of the coherent and homogeneous condensate, which in turn is coupled to the Friedmann equations for the scale factor.

In the case of Bose-Einstein condensation the chemical potential attains its maximum value $\mu = m$ in which case the contribution from the particles *outside* the condensate is $I_B[x_d, x_d]$ which has a strong infrared divergence *for any value of the mass*. This is a consequence of the fact that for ultrarelativistic Bose-Condensed particles the distribution function diverges at $p = 0$ *for any value of the mass*.

However, as discussed above this infrared divergence associated with a Bose-Einstein condensate must be carefully assessed in an expanding cosmology because infrared modes with wavelengths larger than the size of the

horizon will not be in LTE since no causal processes can establish thermalization for superhorizon wavelengths. Hence we *conjecture* that the integrals of the Bose-Einstein distribution function must be cutoff in the infrared at a momentum of the order of the Hubble scale at decoupling, $H_d \sim \sqrt{g_d} T_d^2 / M_{Pl}$. Such a cutoff leads to the estimate

$$I_{BEC} \sim \frac{m M_{Pl}}{\sqrt{g_d} T_d^2} \sim \frac{10^{12}}{\sqrt{g_d}} \left(\frac{m}{\text{keV}} \right) \left(\frac{\text{GeV}}{T_d} \right)^2. \quad (3.38)$$

If this were the *only* DM component, the resulting comoving free-streaming length at $z = 0$ (*today*) is given by

$$\lambda_{fs}(0) \sim \frac{0.014 \text{ pc}}{g_d^{\frac{1}{12}}} \left(\frac{\text{keV}}{m} \right)^{\frac{3}{2}} \left(\frac{T_d}{\text{GeV}} \right) \quad (3.39)$$

It is clear from the discussion above that the microphysics of decoupling of light bosonic particles, with or without a Bose-Einstein condensate requires a thorough assessment of the infrared behavior of the distribution function. The primordial velocity dispersion is very sensitive to this cutoff whose origin lies in the causal aspects of decoupling. This important physical aspect must be studied in deeper detail, a task that is certainly beyond the realm of this article, but we can nevertheless conclude that a light bosonic particle decoupled in LTE can effectively act as CDM as a consequence of the infrared sensitivity of the moment $\langle 1/\vec{V}^2 \rangle$ for Bosonic thermal relics either condensed or not.

- **Non-relativistic particles:** The distribution function after freeze out in LTE is the Maxwell-Boltzmann distribution

$$f_d^{(0)}(p_c) = n_d \left[\frac{m T_d}{2\pi} \right]^{-\frac{3}{2}} e^{-\frac{p_c^2}{2m T_d}} = n_d \left[\frac{m T_d}{2\pi} \right]^{-\frac{3}{2}} e^{-\frac{y^2 T_d}{2m}} \quad (3.40)$$

where n_d is the number of particles per comoving volume at freeze-out[18]

$$n_d = \frac{2\pi^2}{45} g_d T_d^3 Y_\infty \quad (3.41)$$

and Y_∞ is obtained from the solution of the kinetic equation and is a function of the annihilation cross section (see section 5.2 in ref.[18]). We find

$$\int_0^\infty f_d^{(0)}(y) dy = \frac{4\pi^4 g_d Y_\infty}{45 x_d} \quad (3.42)$$

and for the integral I_a eqn. (3.15) denoted by I_{NR} for the non-relativistic (Maxwell-Boltzmann) distribution we obtain

$$I_{NR} = \frac{T_d}{m}. \quad (3.43)$$

In the case of (WIMPs) with[35], $m \sim 100 \text{ GeV}$; $T_d \sim 10 \text{ MeV}$ as candidates for cold dark matter, $x_d \sim 10^{-4}$. If this is the *only* DM candidate with $\nu = 1$ with vanishing abundance of the other WDM or HDM candidates, the comoving free streaming length at $z = 0$ is given by

$$\lambda_{fs}(0) \sim \frac{0.014 \text{ pc}}{g_d^{\frac{1}{3}}} \left(\frac{100 \text{ GeV}}{m} \right)^{\frac{1}{2}} \left(\frac{10 \text{ MeV}}{T_d} \right)^{\frac{1}{2}}, \quad (3.44)$$

from which it follows that for WIMPs with $m \sim 100 \text{ GeV}$ that decouple kinetically at $T_d \sim 10 \text{ MeV}$ [35] when $g_d \sim 10$ [18]

$$\lambda_{fs}(0) \sim 6.5 \times 10^{-3} \text{ pc}. \quad (3.45)$$

C. Decoupling out of LTE

In ref.[34] the following distribution function for particles that decouple *out of LTE* and that effectively models several cases of cosmological relevance was introduced,

$$f_d(y) = f_0 f_{eq}\left(\frac{y}{\eta}\right) \theta(y_0 - y), \quad (3.46)$$

where $f_{eq}(\frac{p_c}{\eta T_d})$ is the equilibrium distribution function for a relativistic particle at an effective temperature ηT_d . This form is motivated by detailed studies of production[39] and thermalization process that proceeds by energy transfer from long to short wavelengths via a cascade with a *front* that moves towards the ultraviolet[40]. If the interaction rate for mode mixing becomes smaller than the expansion rate the advance of this front is *interrupted* at a fixed value of the momentum, identified here to be $p_c^0 = y_0 T_d$ where T_d is the temperature of the environmental degrees of freedom that are in LTE at the time of decoupling[34]. The amplitude f_0 and effective temperature $\eta T_d \leq T_d$ reflect an incomplete thermalization behind the front of the cascade and determine the average number of particles in its *wake*[40]. The non-equilibrium distribution function (3.46) yields a fairly accurate description of these processes and the decoupling out of LTE.

Remarkably, this non-LTE distribution function also describes[34] sterile neutrinos produced non-resonantly via the Dodelson-Widrow[32] (DW) mechanism or resonantly via a lepton-driven MSW resonance[33].

For the general form (3.46) of the distribution function we find

$$I_a = \frac{1}{\eta^2} H\left[\frac{p_c^0}{\eta T_d}\right] ; \quad H[s] = \frac{\int_0^s f_{eq}(y) dy}{\int_0^s y^2 f_{eq}(y) dy}. \quad (3.47)$$

The function $H(s)$ for $f_{eq}(y) = 1/(e^y + 1)$ is a monotonically decreasing function of s with limiting behavior $H(s) \sim 3/s^2$ for $s \rightarrow 0$ and $H(s) \rightarrow 2 \ln(2)/3\zeta(3)$ for $s \rightarrow \infty$, it is displayed in fig. (2) in the interval $0.25 \leq s \leq 2$.

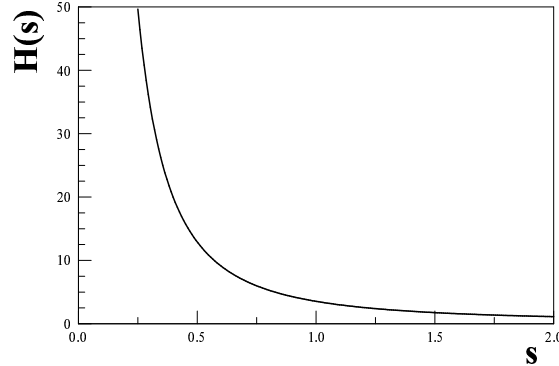


FIG. 2: $H(s)$ vs s for $f_{eq}(y) = 1/(e^y + 1)$.

Although a more detailed assessment of the production and pathway towards thermalization of sterile neutrinos is still being debated[41, 42] we will adopt the semi-phenomenological results of refs.[32, 33] as guiding models for the distribution functions of sterile neutrinos decoupled out of LTE.

- **Sterile neutrinos produced via the (DW) mechanism**[32]: for this case $\eta = 1; s \rightarrow \infty; f_0 \sim 0.043[\text{keV}/m]$ [32, 34]. The integral I_a does not depend on the amplitude f_0 (only the fractional abundance ν is affected by f_0) and is therefore given by the value for an ultrarelativistic fermion with vanishing chemical potential decoupled in LTE[32], namely $I_{DW} = 2 \ln(2)/3\zeta(3) \sim 0.3844$. If these sterile neutrinos are the *only* DM component, namely $\nu = 1$, the free streaming length today is given by the same form as for a Fermionic thermal relic eqn. (3.29) but with vanishing chemical potential,

$$\lambda_{fs}(0) = \frac{22.7}{g_d^{\frac{1}{3}}} \left(\frac{\text{keV}}{m} \right) \text{kpc}. \quad (3.48)$$

In the Dodelson-Widrow scenario[32] the sterile neutrino production rate peaks at $T \sim 130$ MeV which is near the QCD scale. Taking the decoupling temperature in this range results in $g_d \sim 30$ [18], leading to

$$\lambda_{fs}(0) \simeq 7 \text{ kpc} \left(\frac{\text{keV}}{m} \right). \quad (3.49)$$

Therefore $m \sim \text{keV}$ sterile neutrinos produced via the (DW) mechanism yield free streaming lengths today of the order of $\lambda_{fs} \sim 7 \text{ kpc}$. As discussed in ref.[32] there is a potential ambiguity in g_d because near the QCD phase transition there is an abrupt change in the effective number of relativistic degrees of freedom. However, because the cube-root of g_d enters in λ_{fs} the free streaming length is not very sensitive to this ambiguity.

- **Lepton-driven resonantly produced sterile neutrinos**[33]: for this case $\eta = 1, s \sim 0.7; f_0 = 1$ [33, 34], and we find $I_a = H(0.7) = 6.814$. If this species is the *only* DM component we find the free-streaming scale *today*

$$\lambda_{fs}(0) = \frac{5.4}{g_d^{\frac{1}{3}}} \left(\frac{\text{keV}}{m} \right) \text{ kpc}. \quad (3.50)$$

The production rate in the resonant case also seems to peak near the QCD scale[33] at which $g_d \sim 30$. Taking this value for the decoupling temperature we obtain the estimate

$$\lambda_{fs}(0) = 1.73 \text{ kpc} \left(\frac{\text{keV}}{m} \right). \quad (3.51)$$

Hence $m \sim \text{keV}$ sterile neutrinos produced via the lepton-driven resonant mechanism[33] mechanism yield free streaming lengths today of the order of $\lambda_{fs} \sim 2 \text{ kpc}$, which is consistent with the values $\sim 3 \text{ kpc}$ used in[19] to study the first stars forming in filamentary structures.

The enhancement of I_a and consequently the shortening of the free streaming scale in the case of sterile neutrinos produced resonantly out of LTE follows from the fact that the distribution function resulting from the resonant production mechanism favors low momenta. Therefore for the same value of the contribution of sterile neutrinos produced resonantly or non-resonantly out of LTE, the free streaming length in the case of resonance production is ~ 4 times *smaller* than either a thermal Fermionic relic with vanishing chemical potential or a sterile neutrino produced non-resonantly via the (DW) mechanism. Just as in the (DW) scenario, there is an ambiguity in the precise determination of g_d near the QCD scale, but again the free streaming length is not very sensitive to this ambiguity because of the power $1/3$.

Although the details of the mechanism of production and decoupling of sterile neutrinos are still under scrutiny[41], we take the above results as a guideline. In particular the lesson from the resonant production case highlights the potentially dramatic reduction of the free streaming length when the non-equilibrium distribution function favors smaller values of the momentum. This observation clearly calls for a deeper assessment of the production and thermalization of sterile neutrinos to extract reliably their non-equilibrium distribution function.

Combining all the results above into eqns. (3.24,3.25), the free streaming length for a *mixture* of Fermionic and Bosonic thermal relics, including a possible BEC, WIMPs and sterile neutrinos produced non-resonantly or resonantly assumed to be described by the respective distribution functions quoted in refs.[32, 33] is given by

$$\frac{1}{\lambda_{fs}^2(z)} = \frac{1}{(1+z)} \left[\frac{0.071}{\text{kpc}} \right]^2 \sum_{\text{species}} \left\{ \nu_F g_{d,F}^{\frac{2}{3}} \left(\frac{m_F}{\text{keV}} \right)^2 I_F[u] + \nu_s g_{d,s}^{\frac{2}{3}} \left(\frac{m_s}{\text{keV}} \right)^2 6.814 + \nu_B g_{d,B}^{\frac{2}{3}} \left(\frac{m_B}{\text{keV}} \right)^2 I_B[x_d, u_d] + 10^{12} \nu_{wimp} g_{d,wimp}^{\frac{2}{3}} \left(\frac{m_{wimp}}{100 \text{ GeV}} \right) \left(\frac{T_d}{10 \text{ MeV}} \right) \right\}, \quad (3.52)$$

where the index F refers to thermal Fermions and sterile neutrinos produced non-resonantly via the (DW) mechanism[32] for which $I_F[0] = 2 \ln(2)/3\zeta(3) = 0.3844$, and the label s refers solely to sterile neutrinos produced resonantly via the mechanism in ref.[33]. In each case $g_{d,a}$ is the effective number of ultrarelativistic degrees of freedom when the corresponding species decouples. For sterile neutrinos produced by either mechanism $g_d \sim 30$ corresponding to a decoupling temperature near the QCD scale.

IV. CONCLUSIONS AND FURTHER QUESTIONS

In this article we have implemented methods from the theory of multicomponent plasmas to study free streaming of a *mixture* of non-relativistic DM candidates that include Fermionic and Bosonic particles that decouple in LTE while relativistic, including the possibility of a Bose-Einstein condensate, heavy thermal relics that decoupled in when non-relativistic (WIMPs), and sterile neutrinos that decouple *out of LTE* when they are relativistic. We solve *exactly* the Boltzmann-Vlasov equation for the gravitational perturbations in a non-expanding cosmology and obtain the “gravitational polarization function” whose zeroes determine the dispersion relations of the collective excitations of the self-gravitating collisionless gas of particles. The free-streaming wave vector is obtained from the marginal solution that separates Landau damped short wavelength perturbations from unstable collective modes. We obtain the free-streaming length for *arbitrary* (but isotropic) distributions of the particles that decoupled *in or out of LTE* solely in terms of the fractional abundance of the different species and integrals of their distribution functions which depend on the microphysics at decoupling.

Because all of the components are non-relativistic and assuming that the expansion is slow we provide an adiabaticity argument that allows us to implement a simple rescaling of densities, velocities dispersion and wavelengths to extract the redshift dependence of the free-streaming length. The validity of this adiabatic approximation is confirmed by the similarity of the result to the Jeans’ length for Newtonian perturbations in an expanding cosmology and by the explicit result for the free-streaming wavevector for thermal neutrinos obtained in ref.[29].

The main result for the free streaming length as a function of redshift for an arbitrary mixture of DM components is

$$\frac{1}{\lambda_{fs}^2(z)} = \frac{1}{(1+z)} \left[\frac{0.071}{\text{kpc}} \right]^2 \sum_a \nu_a g_{d,a}^{\frac{2}{3}} \left(\frac{m_a}{\text{keV}} \right)^2 I_a, \quad (4.1)$$

where

$$I_a = \left[\frac{\int_0^\infty f_{d,a}^{(0)}(y) dy}{\int_0^\infty y^2 f_{d,a}^{(0)}(y) dy} \right] \quad (4.2)$$

is a ratio of integrals of the distribution functions that only depends on the $m_a/T_d, \mu_a/T_d$ and dimensionless couplings and is completely determined by the microphysics at decoupling. Evaluating these integrals for thermal Fermionic and Bosonic relics (with or without condensation), WIMPs and sterile neutrinos decoupled out of LTE either resonantly or non-resonantly with the distribution functions obtained in refs.[32, 33] respectively, we find the general result

$$\begin{aligned} \frac{1}{\lambda_{fs}^2(z)} = \frac{1}{(1+z)} \left[\frac{0.071}{\text{kpc}} \right]^2 \sum_{\text{species}} \left\{ \nu_F g_{d,F}^{\frac{2}{3}} \left(\frac{m_F}{\text{keV}} \right)^2 I_F[u] + \nu_s g_{d,s}^{\frac{2}{3}} \left(\frac{m_s}{\text{keV}} \right)^2 6.814 + \nu_B g_{d,B}^{\frac{2}{3}} \left(\frac{m_B}{\text{keV}} \right)^2 I_B[x_d, u_d] + \right. \\ \left. 10^{12} \nu_{wimp} g_{d,wimp}^{\frac{2}{3}} \left(\frac{m_{wimp}}{100 \text{ GeV}} \right) \left(\frac{T_d}{10 \text{ MeV}} \right) \right\}, \end{aligned} \quad (4.3)$$

where ν_a is the partial fraction of each component with $\sum_a \nu_a = 1$, the sum over Fermionic species (F) includes thermal relics *and* sterile neutrinos produced non-resonantly via the (DW) mechanism[32], for which the chemical potential vanishes and $I_F[0] = 0.3844$, the label s is for sterile neutrinos produced via a lepton-driven (MSW) resonance as described in ref.[33], and the label B stands for condensed or non-condensed Bosonic thermal relics.

This expression features several important consequences relevant for large scale structure formation:

- A non-negligible $\nu_{wimp} \gg 10^{-12}$, for a CDM candidate with $m \sim 100 \text{ GeV}$ which decoupled at $T_d \sim 10 \text{ MeV}$ [35] overwhelms all other components (but for a possible BEC) and leads to small free streaming lengths consistent with CDM regardless of the presence of WDM or HDM components. For $\nu_{wimp} \gg 10^{-12}$ the free-streaming length of *mixed* DM is completely dominated by WIMPs and is given today by

$$\lambda_{fs}(0) \sim \frac{0.014 \text{ pc}}{g_d^{\frac{1}{3}}} \left(\frac{100 \text{ GeV}}{m} \right)^{\frac{1}{2}} \left(\frac{10 \text{ MeV}}{T_d} \right)^{\frac{1}{2}}, \quad (4.4)$$

from which it follows that for WIMPs with $m \sim 100 \text{ GeV}$ that decouple kinetically at $T_d \sim 10 \text{ MeV}$ [35] when $g_d \sim 10$ [18]

$$\lambda_{fs}(0) \sim 6.5 \times 10^{-3} \text{ pc}. \quad (4.5)$$

This cut-off scale might well be related to the smallest non-linear structures found in[10] unless there is some substantial violent relaxation and merging.

- For vanishing chemical potential $u_d = 0$, non-BEC Bosonic ultrarelativistic relics feature an infrared enhancement in $I_B[x_d, 0]$ which must be regulated by assuming that the integrals of the distribution function are cutoff of order of the Hubble scale at decoupling $H_d \sim \sqrt{g_d} T_d^2 / M_{Pl}$. If this is the *only* DM component, the free-streaming length today is

$$\lambda_{fs}(0) \sim \frac{14}{g_d^{\frac{1}{3}} \sqrt{I_B[x_d, 0]}} \left(\frac{\text{keV}}{m} \right) \text{kpc} \quad ; \quad I_B[x_d, 0] \sim \frac{1}{2\zeta(3)} \ln \left[\frac{2T_d}{H_d + \sqrt{H_d^2 + m^2}} \right]. \quad (4.6)$$

For BEC Bosons $x_d = u_d$ the function $I_B[x_d, x_d]$ is divergent as a consequence of the infrared divergence in the numerator of I_a (eq. 3.15). However, we highlighted that the physics of BEC formation in an expanding cosmology must be assessed in greater detail in order to understand the behavior of superhorizon modes. This observation also holds for the non-condensed Bose gas decoupling when relativistic with $\mu/T_d \ll 1$ because the distribution function in this case is also infrared sensitive. In both these cases an estimate of the free streaming length may be obtained by introducing an infrared cutoff in the integral in the numerator of I_a of the order of the Hubble scale, since superhorizon modes cannot establish thermal equilibrium via causal processes as discussed above. If a BEC is the *only* DM component, its free-streaming length today is approximately given by

$$\lambda_{fs}(0) \sim \frac{0.014 \text{ pc}}{g_d^{\frac{1}{12}}} \left(\frac{\text{keV}}{m} \right)^{\frac{3}{2}} \left(\frac{T_d}{\text{GeV}} \right). \quad (4.7)$$

Therefore even when these relics decoupled when they were ultrarelativistic, they could effectively act as CDM components. This is a consequence of the fact that the distribution functions favor small momenta.

- If sterile neutrinos that decouple *out of LTE* near the QCD scale produced either non-resonantly via the (DW)[32] mechanism or via a lepton-driven MSW resonance[33] near the QCD scale are the *only* DM components we find the following free-streaming lengths today

$$\lambda_{fs}(0) \simeq 7 \text{ kpc} \left(\frac{\text{keV}}{m} \right) \quad \text{non - resonant} \quad (4.8)$$

$$\lambda_{fs}(0) \simeq 1.73 \text{ kpc} \left(\frac{\text{keV}}{m} \right) \quad \text{resonant}. \quad (4.9)$$

The smaller values of the free streaming length are compatible with those in ref.[19] where it is found that first stars form in filamentary structures with length scales of the order of the free streaming scale and within a factor $\sim 3 - 4$ seem to be also consistent with the “cores” resulting from the fit of the density profile for dwarf spheroidal galaxies in ref.[14] $\sim 0.5 \text{ kpc}$. The larger values are consistent with those found in ref.[15] for the cored profiles of spiral galaxies $\sim 5 - 10 \text{ kpc}$.

We believe that these results lead to a significant advance in the understanding of collisionless (DM) because trying to obtain a reliable estimate of the free-streaming lengths via the numerical integration of Gilbert’s equations for a combination of *arbitrary* distribution functions corresponding to particles that decoupled in or out of LTE is undoubtedly a daunting task.

The sensitivity of the free-streaming scale to the details of the distribution function at low momentum and the importance of a reliable determination of the free-streaming length as a measure of the cutoff of the power spectrum of linearized cosmological perturbations require a fundamentally sound understanding of the microphysics of production and decoupling of sterile neutrinos, a program currently underway[41].

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